# Chow Motives

Idea of Grothendieck: Universal cohomology theory.

Let $k$ be a field and $\mathrm{Var}(k)$ be the category of smooth proj. varieties $/k$. A <u>nice</u> cohomology theory is a contravariant functor $H^\bullet : \mathrm{Var}(k) \longrightarrow \{\text{Graded f.d. vector spaces}/k\}$, that has good properties (such as a Weil cohomology theory).

<u>Example</u>

1) $X \rightsquigarrow H^\bullet_{et}(\overline{X}, \mathbb{Q}_\ell)$
2) $X/_{\mathbb{C}} \rightsquigarrow H^\bullet_{dR}(X)$.

Grothendieck: there should be a "universal" cohomology theory $h : \mathrm{Var}(k) \xrightarrow{\text{contra}} A$, where $A$ is at least an additive category, so that all other cohomology functors factor through $h$.

Take $f \in \mathrm{Hom}_{\mathrm{Var}(k)}(X, Y)$ and $\Gamma_f$ be its graph in $X \times Y$. We know $f^* : H^\bullet(Y) \to H^\bullet(X)$ is equal to $p_*(cl_{X\times Y}(\Gamma_f) \cdot q^*(-))$, where $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$. Second take: why <u>just</u> $\Gamma_f$?

If $Z \subset X \times Y$ is any subvariety, it defines a map $Z^*(-) = p_*(cl_{X\times Y}(Z) \cdot q^*(-))$. If $Z \underset{\text{rat.}}{\sim} Z'$ then $Z^* = Z'^*$. These are called correspondences.

<u>Def</u>: $\mathrm{Chow}(k)$ = Category of Chow Motives.

objects = smooth proj. varieties $/k$.

morphisms = subvarieties of $X \times Y$ upto $\mathbb{Q}$-equiv. $= A^\bullet(X \times Y)$.

↳ composition: $W \subset Z \times Y$, $S \subset Y \times X$, then

$$W \circ Z = p_{XZ*}(p_{XZ}^* Z \cdot p_{YZ}^* W)$$

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "p_{XY}"'] \arrow[d, "p_{XZ}"] \arrow[dr, "p_{YZ}"] & \\ X \times Y & X \times Z & Y \times Z \end{tikzcd}$$

We leave associativity and $\Delta \subset X \times X$ is the identity as an exercise. There are a few variations of this theme, no one has settled on the "right" one. We have a functor: $h : \mathrm{Var}(k) \longrightarrow \mathrm{Chow}(k)$ by $(f : X \to Y) \longmapsto (\Gamma_f \subset X \times Y)$. Can check this is a "universal" functor and morally $\mathrm{Chow}(k)$ is the additivization of $\mathrm{Var}(k)$.

<u>Variants</u>

- $\mathrm{Chow}^\circ(k)$. Morphisms are $\dim Y$ − codimension.
- Can use coarser equiv.
  1) Homological equiv. $cl(Z) = cl(Z') \rightsquigarrow HC(k)$ homological motives.
  
  $\Downarrow$
  
  2) Numerical equiv. $Z \cdot W = Z' \cdot W \;\; \forall W$ appropriate $\rightsquigarrow NC(k)$ numerical motives.

<u>Big Conjecture</u>: Homological + Numerical motives are equivalent.

Note $\mathrm{Chow}(k)$ is additive, but is not abelian. We at least want kernels of projectors: $p \in \mathrm{Hom}(X, X)$, $p^2 = p$ is a projector. Hence $id - p$ is a projector. We then want to split our variety up via Karoubi completion.

<u>Def</u>: A category is pseudo-abelian if it is additive and

1) all projectors have kernels (equalizers)
2) $\mathrm{Ker}\, p \oplus \mathrm{Ker}(1-p) \xrightarrow{\sim} X$ unique.

Given an additive category $D$ its pseudo-abelian completion $\tilde{D}$ is: ob $= \{(X,p) \mid X \in \text{ob}\, D,\ p: X \to X \text{ -projector}\}$ and $\text{Hom}((X,p),(Y,q)) = q \cdot \text{Hom}_D(X,Y) \cdot p$. This gives a pseudo-abelian category, and $i: D \longrightarrow \tilde{D}$, $X \to (X, \text{id})$ is universal among functors from $D$ into pseudo-abelian categories.

We then have $\widetilde{\text{Chow}(k)}$ and $\tilde{X} = (X, \text{id}_X)$.

<u>Corollary</u>: In $\widetilde{\text{Chow}(k)}$, $\tilde{\mathbb{P}}^1 = (\mathbb{P}^1, p_0) \oplus (\mathbb{P}^1, p_1)$, where $p_0 = e \times 1$, $p_1 = 1 \times e \in A^1(\mathbb{P}^1 \times \mathbb{P}^1)$.

We call $\tilde{e} = (\mathbb{P}^1, p_0) = \mathbb{L}$ the Tate motive. We also have a multiplication $(X,Y) \mapsto X \times Y$ (tensor category?).

<u>Fact</u>: $\tilde{\mathbb{P}}^n = \tilde{e} \oplus \mathbb{L} \oplus \mathbb{L}^2 \oplus \cdots \oplus \mathbb{L}^n$ (in a sense $\mathbb{L} \leftrightarrow \mathbb{A}^1_k$, they have the same cohomology).
$\uparrow \mathbb{L} \otimes \mathbb{L}$.

Two steps: we want to invert $\mathbb{L}$ (think of it as $\otimes$'ing w/ a 1-D vector space). And give rational coefficients. We get $\text{Mot}(k)$, the category of motives (chow), $\text{Mot}(k)_{\mathbb{Q}} = \widetilde{\text{Chow}}(k)[\mathbb{L}^{-1}]_{\mathbb{Q}}$.
Call numerical motives $N\text{Mot}(k)$.

<u>Standard Conjectures on Algebraic Cycles</u>: Let $X/\mathbb{C}$ be smooth proj. of dim $n$. Choose an ample $\mathcal{L} \in \text{Pic}\, X$, and $c_1(\mathcal{L}) \in H^2(X, \mathbb{C})$. Define $L: H^i(X) \to H^{i+2}(X)$ by $\alpha \mapsto \alpha \cup c_1(\mathcal{L})$.

<u>Thm: (Hard Lefschetz)</u> $H^{n-i} \xrightarrow{L^i} H^{n+i}$ is an isomorphism $\forall i$. Hence $\exists\, \Lambda: H^j \to H^{j-2}$ such that $\Lambda^i: H^{n+i} \xrightarrow{\sim} H^{n-i}$.
Moreover, $\exists\, h|_{H^j}$ which is multiplication by $(j-n)$, and $(L, \Lambda, h)$ are an $sl_2$-triple on $H^\bullet(X)$.

Now over any field $k$, $H$ a Weil cohomology theory;

<u>Def</u>: $AH^\bullet = \mathbb{Q}$-span of classes $cl_X(-) \in H^\bullet(X)$.

Now $L: AH^i \longrightarrow AH^{i+2}$ as the operator is algebraic (here $L$ is given by the correspondence $\mathcal{L} = \mathcal{L}(\Delta)$).

$$\left.\begin{array}{l} \underline{\text{Conjecture A}}: L^i: AH^{n-i}(X) \xrightarrow{\sim} AH^{n+i}(X) \\ \underline{\text{Conjecture B}}: \Lambda \text{ is algebraic} \\ \underline{\text{Conjecture C}}: \pi^i: H^\bullet(X) \longrightarrow H^i(X) \text{ are algebraic.} \end{array}\right\} X \ \underline{\text{fixed}}.$$

Fact: "$A \Leftrightarrow B \Leftrightarrow C$"

<u>Thm</u>: 1) $B$ holds for all $L$ if it holds for one.
2) $B$ is stable under products, hyperplane sections.
3) $B$ holds for curves, surfaces, abelian varieties, and generalized flag varieties.
4) $C$ holds if $k = \mathbb{F}_q$.

<u>Hodge Standard Conjecture</u>: Define $P^i = \text{Ker}(L^{n-i}: H^i \to H^{2n-i})$. Then $\forall i \le n$, the $\mathbb{Q}$-valued pairing on $AH^{2i} \cap P^{2i}$ $(x,y) \longmapsto (-1)^i \langle L^{n-2i}(x) \cdot y \rangle$ is positive definite.

If char $k=0$, then this holds by Hodge theory.

<u>Conjecture D</u>: If a cycle on $X$ is numerically equivalent to zero, then its homologically equivalent to zero.

Note the Hodge conjecture $\Rightarrow$ ($A \Leftrightarrow D$). Over $\mathbb{C}$, the "usual" Hodge conjecture implies all of the above.

Lemma: $\mathrm{Num}^i(X)$ is a f.g. abelian group.

Thm: Assume $B(X)$ and $\mathrm{Hdg}(X \times X)$. Then
1) The $\mathbb{Q}$-algebra $\mathrm{End}_{NMot(k)}(X)$ is semisimple, hence a product of matrix algebras.
   ($\Rightarrow$ $NMot(k)$ is abelian and semisimple Holy shit)
2) Assume $X/\mathbb{F}_q$, $\phi : \overline{X} \to \overline{X}$ frob. Then $\phi^* H^i(X)$ is semisimple char poly has $\mathbb{Z}$-coeff's, indep. of $H^i(X)$ + eigs have absolute value $q^{i/2}$.

Thm: (Jantzen) $NMot(k)$ is abelian and semisimple (indep of std. conj.) !

Motivic Zeta Function: See Lunt's paper.